

# A CANONICAL RAMSEY THEOREM FOR EXACTLY $m$ -COLOURED COMPLETE SUBGRAPHS

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**ABSTRACT.** Given a colouring of the edges of a graph, we say that the graph is exactly  $m$ -coloured if the number of distinct colours used by the colouring is  $m$ . The question of finding exactly  $m$ -coloured complete subgraphs was first considered by Erickson in 1994; Stacey and Weidl, in 1999, partially settled a conjecture made by Erickson and considered several related problems. The primary aim of this paper is to answer a question posed by Stacey and Weidl. We consider edge colourings of the complete graph on  $\mathbb{N}$  with infinitely many colours, and study under which circumstances we can find exactly  $m$ -coloured complete subgraphs for every natural number  $m$ . We prove that if, for some natural number  $m$ , one cannot find an exactly  $m$ -coloured complete subgraph, then the colouring must admit a complete infinite subgraph which is coloured in one of exactly two specific ways. The techniques that we develop also enable us to resolve some further questions about finding exactly  $m$ -coloured complete subgraphs in colourings with finitely many colours.

## 1. INTRODUCTION

A classical result of Ramsey [9] says that when the edges of a complete graph on a countably infinite vertex set are finitely coloured, one can always find a complete infinite subgraph all of whose edges have the same colour.

Ramsey's Theorem has since been generalised in many ways; most of these generalisations are concerned with finding monochromatic substructures in various coloured structures. For a survey of many of these generalisations, see the book of Graham, Rothschild and Spencer [3]. Ramsey theory has witnessed many developments over the last fifty years and continues to be an area of active research today; see for example [4], [5], [6], [7].

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While one is usually concerned with finding monochromatic substructures in various finitely coloured structures, two alternative directions are as follows. First, one could study colourings that use infinitely many colours, as was first done by Erdős and Rado [1] and by many others after them. Second, one could look for structures which are coloured with exactly  $m$  colours for some  $m \geq 2$ . This was first considered by Erickson [2] and then investigated further by Stacey and Weidl [10]. In this paper, we shall consider the question of finding structures coloured with exactly  $m$  colours in colourings that use infinitely many colours.

The rest of this paper is organised as follows. In Section 2, we present the relevant definitions that we require and the statements of our results. Section 3 is devoted to the proof of our main result. In Section 4, we describe an extension of our main result and some applications of this extension. We conclude the paper in Section 5 with some open problems.

## 2. OUR RESULTS

For a set  $X$ , denote by  $X^{(2)}$  the set of all unordered pairs of elements of  $X$ ; equivalently,  $X^{(2)}$  is the complete graph on the vertex set  $X$ . As always,  $[n]$  will denote  $\{1, \dots, n\}$ , the set of the first  $n$  natural numbers. By a colouring of a graph  $G$ , we will always mean a colouring of the edges of  $G$ .

Let  $\Delta : \mathbb{N}^{(2)} \rightarrow \mathbb{N}$  be a colouring of the edges of the complete graph on  $\mathbb{N}$  with an arbitrary set of colours. Since  $\mathbb{N}^{(2)}$  is countable, we lose no generality in assuming that the set of colours is  $\mathbb{N}$ . We say that a subset  $X$  of  $\mathbb{N}$  is (*exactly*)  $m$ -coloured if  $\Delta(X^{(2)})$ , the set of values attained by  $\Delta$  on the edges with both endpoints in  $X$ , has size exactly  $m$ . Further, we say that  $X \subset \mathbb{N}$  is *rainbow coloured* if no two edges with both endpoints in  $X$  receive the same colour.

Canonical Ramsey theory, which originates in a classical paper of Erdős and Rado [1], provides results about colourings which use an arbitrary set of colours. We shall need the following basic result from their paper. To state their result, we need the notion of left and right coloured sets. We say that  $X \subset \mathbb{N}$  is *left coloured* if for  $i, j, k, l \in X$  with  $i < j$  and  $k < l$ ,  $\Delta(ij) = \Delta(kl)$  if and only if  $i = k$ ; the definition of a *right coloured* set is analogous.

**Theorem 1.** *For every colouring  $\Delta : \mathbb{N}^{(2)} \rightarrow \mathbb{N}$ , there exists an infinite subset  $X$  of  $\mathbb{N}$  such that either*

- $X$  is rainbow coloured; or
- $X$  is 1-coloured; or
- $X$  is left coloured; or
- $X$  is right coloured.

For a colouring  $\Delta : \mathbb{N}^{(2)} \rightarrow \mathbb{N}$  of the complete graph on  $\mathbb{N}$  with an arbitrary set of colours, we define the set

$$\mathcal{G}_\Delta := \{m \in \mathbb{N} : \exists X \subset \mathbb{N} \text{ such that } X \text{ is } m\text{-coloured}\}.$$

In this note, we will mainly be concerned with finding  $m$ -coloured complete subgraphs in colourings that use infinitely many colours. We call a colouring  $\Delta : \mathbb{N}^{(2)} \rightarrow \mathbb{N}$  an *infinite colouring* if it is a colouring that uses infinitely many colours; in other words, if the image of  $\Delta$  is infinite. Given such an infinite colouring  $\Delta : \mathbb{N}^{(2)} \rightarrow \mathbb{N}$ , our aim in this paper is to study the set  $\mathcal{G}_\Delta$ .

Stacey and Weidl [10] considered the question of which natural numbers  $m$  are guaranteed to be elements of  $\mathcal{G}_\Delta$  for every infinite colouring  $\Delta$ . By considering an infinite rainbow colouring  $\Delta$  of  $\mathbb{N}$ , we see that unless  $m = \binom{n}{2}$  for some  $n \geq 2$ ,  $m$  is not guaranteed to be a member of  $\mathcal{G}_\Delta$ . In the other direction, since an edge is a 1-coloured complete graph,  $\binom{2}{2} = 1$  is always an element of  $\mathcal{G}_\Delta$ . Stacey and Weidl were able to show that  $\binom{3}{2} = 3$  is also always an element of  $\mathcal{G}_\Delta$  for every infinite colouring  $\Delta$ . But for  $n \geq 4$ , they were unable to decide whether or not there exists an infinite colouring  $\Delta$  such that  $\binom{n}{2} \notin \mathcal{G}_\Delta$ . In particular, they asked if all natural numbers of the form  $\binom{n}{2}$  must be contained in  $\mathcal{G}_\Delta$  for every infinite colouring  $\Delta$ .

Here, we shall consider a more general question: when is  $\mathcal{G}_\Delta \neq \mathbb{N}$ ? As remarked above, for an injective colouring  $\Delta$  which rainbow colours  $\mathbb{N}$ ,  $\mathcal{G}_\Delta = \{\binom{n}{2} : n \geq 2\} \neq \mathbb{N}$ . There is another infinite colouring  $\Delta$  for which  $\mathcal{G}_\Delta \neq \mathbb{N}$  which is slightly less obvious than the rainbow colouring. Given  $X \subset \mathbb{N}$ , if there is a vertex  $v \in X$  such that  $X \setminus \{v\}$  is 1-coloured and all the edges between  $v$  and  $X \setminus \{v\}$  have distinct colours (which are also all different from the colour of  $X \setminus \{v\}$ ), then we say that  $X$  is *star coloured (with centre  $v$ )*. It is easy to check (see Figure 1) that if  $\mathbb{N}$  is star coloured by  $\Delta$ , then  $\mathcal{G}_\Delta = \mathbb{N} \setminus \{2\}$ .

Our main result, stated below, is that the two colourings described above are, in a sense, the only colourings for which  $\mathcal{G}_\Delta \neq \mathbb{N}$ .

**Theorem 2.** *For every infinite colouring  $\Delta : \mathbb{N}^{(2)} \rightarrow \mathbb{N}$ , either*

- $\mathcal{G}_\Delta = \mathbb{N}$ ; or
- *there exists a rainbow coloured infinite subset of  $\mathbb{N}$ ; or*
- *there exists a star coloured infinite subset of  $\mathbb{N}$ .*

An immediate consequence of Theorem 2 is that the answer to the question posed by Stacey and Weidl is in the affirmative.

**Corollary 3.** *For every infinite colouring  $\Delta : \mathbb{N}^{(2)} \rightarrow \mathbb{N}$ , and for every natural number  $n \geq 2$ ,  $\binom{n}{2} \in \mathcal{G}_\Delta$ .*

We do not prove Theorem 2 as stated. We instead prove a stronger result which will need the notion of ignoring a designated colour. In Section 3, we shall introduce this notion and prove Theorem 2.

In the context of colourings using finitely many colours, the question of finding  $m$ -coloured complete infinite subgraphs was first considered by Erickson [2]. Let  $\Delta : \mathbb{N}^{(2)} \rightarrow [k]$  be a

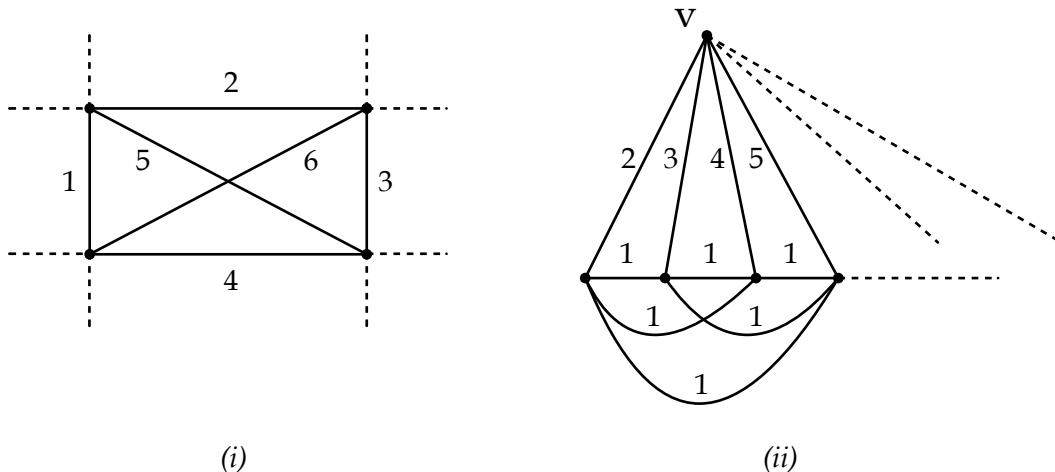


FIGURE 1. (i) A rainbow colouring (ii) A star colouring with centre  $v$

colouring of the edges of the complete graph on the natural numbers with exactly  $k \geq 2$  colours and let

$$\mathcal{F}_\Delta := \{m \in [k] : \exists X \subset \mathbb{N} \text{ such that } X \text{ is infinite and } m\text{-coloured}\}.$$

Clearly,  $k \in \mathcal{F}_\Delta$  as  $\Delta$  is surjective. Ramsey's theorem tells us that  $1 \in \mathcal{F}_\Delta$ . Erickson [2] noted that a fairly straightforward application of Ramsey's theorem enables one to show that  $2 \in \mathcal{F}_\Delta$ . He also conjectured that with the exception of  $1, 2$  and  $k$ , no other elements are guaranteed to be in  $\mathcal{F}_\Delta$  and that if  $k > k' > 2$ , then there is colouring  $\Delta$  with exactly  $k$  colours such that  $k' \notin \mathcal{F}_\Delta$ . Stacey and Weidl [10] settled this conjecture in the case where  $k$  is much bigger than  $k'$ . More precisely, for any  $k' > 2$ , they showed that there is a constant  $C_{k'}$  such that if  $k > C_{k'}$ , then there is an exact  $k$ -colouring  $\Delta$  such that  $k' \notin \mathcal{F}_\Delta$ . Recently, the second author [8] studied how small the set  $\mathcal{F}_\Delta$  can be and showed that for every  $\Delta : \mathbb{N}^{(2)} \rightarrow [k]$ ,  $|\mathcal{F}_\Delta| \geq \sqrt{2k}$ , and that this lower bound is tight for infinitely many values of  $k$ .

Stacey and Weidl [10] also formulated the following proposition  $F(V, k, m)$ :

**$F(V, k, m)$ :** *For every colouring of the complete graph on the vertex set  $V$  with exactly  $k$  colours, there is a complete subgraph which is  $m$ -coloured.*

When  $V$  is finite, note that  $F(V, k, m)$  holds trivially if  $\binom{|V|}{2} < k$  since no colouring  $\Delta$  can then use  $k$  distinct colours. When  $V$  is infinite, say when  $V = \mathbb{N}$ , note that saying the proposition  $F(\mathbb{N}, k, m)$  holds is the same as saying  $m \in \mathcal{G}_\Delta$  for every colouring  $\Delta : \mathbb{N}^{(2)} \rightarrow [k]$  which uses exactly  $k$  colours. The difference between studying the proposition  $F(\mathbb{N}, k, m)$  as opposed

to the set  $\mathcal{F}_\Delta$  is that we only take into account complete *infinite* subgraphs in  $\mathcal{F}_\Delta$ ; we also consider *finite* complete subgraphs when studying  $F(\mathbb{N}, k, m)$ .

Clearly,  $F(V, k, 1)$  always holds. Stacey and Weidl proved that  $F(V, k, 2)$  holds for any  $k \geq 2$  as long as  $V$  is either infinite or a finite set whose size is larger than  $R(k+1; k)$  (the Ramsey number for finding a monochromatic  $K_{k+1}$  when using  $k$  colours). Further, they showed that  $F(V, k, 3)$  holds for any set  $V$  and any  $k \geq 3$ . They raised the question of determining when the proposition  $F(V, k, m)$  holds in general.

In the case where  $V = \mathbb{N}$ , note that the proposition  $F(\mathbb{N}, k, m)$  is false for infinitely many values of  $k$  when  $m$  is not of the form  $\binom{n}{2}$  or  $\binom{n}{2} + 1$  for some natural number  $n \geq 2$ . To see this, suppose that  $k = \binom{l}{2} + 1$  for some  $l \geq 2$ , and consider a colouring of  $\mathbb{N}^{(2)}$  which colours all the edges with both endpoints in  $[l]$  with  $\binom{l}{2}$  distinct colours and all the remaining edges with the one colour that has not been used so far. It is easy to check that for this colouring, every complete subgraph is either  $\binom{n}{2}$ -coloured or  $(\binom{n}{2} + 1)$ -coloured for some  $n \geq 2$ .

On the other hand, when  $m$  is of the form  $\binom{n}{2}$  or  $\binom{n}{2} + 1$  for some natural number  $n \geq 2$ , we have the following positive result.

**Theorem 4.** *For all  $n \in \mathbb{N}$ , there exists a natural number  $C = C(n)$  such that for all  $k \geq C$ , both  $F(\mathbb{N}, k, \binom{n}{2})$  and  $F(\mathbb{N}, k, \binom{n}{2} + 1)$  hold.*

Finally, we consider the question of finding  $m$ -coloured complete bipartite subgraphs in colourings of complete bipartite graphs. We are able to prove the following.

**Theorem 5.** *For all  $m \in \mathbb{N}$ , there exists a natural number  $C' = C'(m)$  such that for every colouring  $\Delta : \mathbb{N}_1 \times \mathbb{N}_2 \rightarrow \mathbb{N}$  of the complete bipartite graph between two copies of  $\mathbb{N}$  that uses at least  $C'$  colours, there exist  $X \subset \mathbb{N}_1$  and  $Y \subset \mathbb{N}_2$  such that the complete bipartite subgraph between  $X$  and  $Y$  is  $m$ -coloured.*

It turns out that the techniques used to prove Theorem 2 also allow us to prove a finitary version of the same theorem. In Section 4, we present this finitary result and use it to prove both Theorems 4 and 5 in slightly stronger forms.

### 3. PROOF OF THE MAIN THEOREM

To prove Theorem 2, we consider colourings  $\Delta : \mathbb{N}^{(2)} \rightarrow \mathbb{N} \cup \{\mathbf{0}\}$  which are allowed to colour edges with a special colour,  $\mathbf{0}$ . The role of the special colour  $\mathbf{0}$  is simple; when looking for  $m$ -coloured complete subgraphs, we do not count the colour  $\mathbf{0}$ . Edges coloured with the colour  $\mathbf{0}$  can be thought of as being absent in the graph or equivalently, as being “colourless”.

Given  $\Delta : \mathbb{N}^{(2)} \rightarrow \mathbb{N} \cup \{\mathbf{0}\}$ , for a subset  $X$  of  $\mathbb{N}$ , we shall write  $\gamma_\Delta(X)$ , or  $\gamma(X)$  in short, for the number of distinct colours in the complete subgraph induced by  $X$  ignoring  $\mathbf{0}$ , that is  $|\Delta(X^{(2)}) \setminus \{\mathbf{0}\}|$ . For disjoint subsets  $X$  and  $Y$ , write  $\gamma(X, Y)$  for the number of distinct

colours in the complete bipartite subgraph between  $X$  and  $Y$  ignoring  $\mathbf{0}$ . For a vertex  $v \in \mathbb{N}$ , we shall write  $\gamma(v)$  for  $\gamma(\{v\}, \mathbb{N} \setminus \{v\})$ , the number of distinct colours of the edges incident on  $v$  ignoring  $\mathbf{0}$ .

We say that a colouring  $\Delta : \mathbb{N}^{(2)} \rightarrow \mathbb{N} \cup \{\mathbf{0}\}$  is a **0-infinite colouring** if the image of  $\Delta$  is infinite. We define the set  $\mathcal{G}_\Delta$  for a **0-infinite colouring**  $\Delta$  in the obvious way by setting

$$\mathcal{G}_\Delta := \{\gamma(X) : X \subset \mathbb{N}\}.$$

Note that a **0-infinite colouring** does not necessarily have to use the colour  $\mathbf{0}$ ; consequently, an infinite colouring  $\Delta : \mathbb{N}^{(2)} \rightarrow \mathbb{N}$  is also a **0-infinite colouring** and in this case, the above definition of the set  $\mathcal{G}_\Delta$  agrees with the earlier definition for infinite colourings.

Since every infinite colouring  $\Delta : \mathbb{N}^{(2)} \rightarrow \mathbb{N}$  is also a **0-infinite colouring**, the following result easily implies Theorem 2.

**Theorem 6.** *For every 0-infinite colouring  $\Delta : \mathbb{N}^{(2)} \rightarrow \mathbb{N} \cup \{\mathbf{0}\}$ , either*

- $\mathcal{G}_\Delta = \mathbb{N}$ ; or
- *there exists a rainbow coloured infinite subset of  $\mathbb{N}$ ; or*
- *there exists a star coloured infinite subset of  $\mathbb{N}$ .*

For any finite set of colours  $\mathcal{C}$ , note that if we alter a **0-infinite colouring** by recolouring all the edges which are coloured with a colour from  $\mathcal{C}$  with the colour  $\mathbf{0}$ , the resulting colouring is still a **0-infinite colouring**. This makes the statement of Theorem 6 more amenable to induction than that of Theorem 2 and motivates the stronger statement of Theorem 6.

If we have a partition  $X = X_1 \sqcup X_2 \sqcup \dots \sqcup X_n$  of a subset  $X$  of  $\mathbb{N}$ , then note that

$$\sum_{1 \leq i \leq n} \gamma(X_i) + \sum_{1 \leq i < j \leq n} \gamma(X_i, X_j) \geq \gamma(X).$$

Consequently, if  $\gamma(X) = \infty$ , then at least one of the terms on the left is infinite; we will make use of this fact repeatedly.

Next, we state a technical lemma about “almost bipartite colourings” which will be useful in proving Theorem 6.

**Lemma 7.** *Suppose a 0-infinite colouring  $\Delta : \mathbb{N}^{(2)} \rightarrow \mathbb{N} \cup \{\mathbf{0}\}$  is such that*

- $\gamma(v) < \infty$  for all  $v \in \mathbb{N}$ , and
- *there is a partition of  $\mathbb{N} = A \sqcup B$  such that  $\gamma(A) < \infty$ ,  $\gamma(B) < \infty$  and  $\gamma(A, B) = \infty$ .*

*Then, for every natural number  $m$ , there exists a subset  $X$  of  $\mathbb{N}$  such that  $X \cap A \neq \emptyset$ ,  $X \cap B \neq \emptyset$  and  $\gamma(X) = m$ .*

Our strategy for proving both Theorem 6 and Lemma 7 is to inductively construct a set  $X$  for which  $\gamma(X) = m$ . To do this, we shall first recolour some edges with the colour  $\mathbf{0}$ . We then

find a set  $X'$  with  $\gamma(X') = m'$  for a suitably chosen  $m' < m$ . Finally, we use the recoloured edges in conjunction with  $X'$  to obtain  $X$ .

We first prove Lemma 7 and then show how to deduce Theorem 6 from it.

**3.1. Proof of Lemma 7.** Before we begin, let us note some consequences of our assumptions about the colouring  $\Delta$ . Since  $\gamma(v) < \infty$  for all  $v \in \mathbb{N}$  and  $\gamma(A, B) = \infty$ , both  $A$  and  $B$  must be infinite. Further, observe that if  $\gamma(U) = \infty$  for some  $U \subset \mathbb{N}$ , then since  $\gamma(A) < \infty$  and  $\gamma(B) < \infty$ , both  $U \cap A$  and  $U \cap B$  must be infinite.

We proceed by induction on  $m$ . The result is trivial for  $m = 1$ . Assuming the result for all  $m' < m$ , we shall prove the result for  $m$ .

Pick an edge  $uv$  of nonzero colour, say  $c$ , such that  $u \in A$  and  $v \in B$ . We know that  $\gamma(u) < \infty$ . We may assume, relabeling nonzero colours if necessary, that the colours of the edges incident on  $u$  are  $0, 1, \dots, \gamma(u)$ . We then partition

$$\mathbb{N} \setminus \{u\} = U_0 \sqcup U_1 \sqcup \dots \sqcup U_{\gamma(u)}$$

where  $U_i$  is the set of all vertices that are joined to  $u$  by an edge of colour  $i$ . If  $\gamma(U_0) < \infty$ , then we have the following three cases.

CASE 1:  $\gamma(U_i) = \infty$  for some  $i \neq 0$

We begin by observing (see Figure 2) that

$$\gamma(U_i \cap A) + \gamma(U_i \cap B) + \gamma(U_i \cap A, U_i \cap B) \geq \gamma(U_i).$$

Since  $\gamma(U_i \cap A) \leq \gamma(A) < \infty$  and  $\gamma(U_i \cap B) \leq \gamma(B) < \infty$ , we conclude that  $\gamma(U_i \cap A, U_i \cap B) = \infty$ . So, if we define a  $0$ -infinite colouring  $\Delta' : U_i^{(2)} \rightarrow \mathbb{N} \cup \{0\}$  by

$$\Delta'(e) = \begin{cases} \Delta(e) & \text{if } \Delta(e) \neq i \\ 0 & \text{if } \Delta(e) = i \end{cases}$$

then, by the induction hypothesis, there exists a subset  $X'$  of  $U_i$  such that  $X' \cap (U_i \cap A) \neq \emptyset$ ,  $X' \cap (U_i \cap B) \neq \emptyset$  and  $\gamma_{\Delta'}(X') = m - 1$ . Observe that all the edges between  $u$  and  $X' \subset U_i$  have colour  $i$ . Since the colour  $i$  is not counted by  $\gamma_{\Delta'}$ , we have  $\gamma_{\Delta}(X' \cup \{u\}) = m$ . Therefore,  $X = X' \cup \{u\}$  is the required subset since  $X \cap A \neq \emptyset$  and  $X \cap B \neq \emptyset$ .  $\square$

CASE 2:  $\gamma(U_i, U_j) = \infty$  for some  $0 < i < j$

Observe (see Figure 3) that  $\gamma(U_i \cap A, U_j \cap A) \leq \gamma(A) < \infty$  and  $\gamma(U_i \cap B, U_j \cap B) \leq \gamma(B) < \infty$ . So we must either have  $\gamma(U_i \cap A, U_j \cap B) = \infty$  or  $\gamma(U_i \cap B, U_j \cap A) = \infty$ . Without loss of generality, assume that  $\gamma(U_i \cap A, U_j \cap B) = \infty$ .

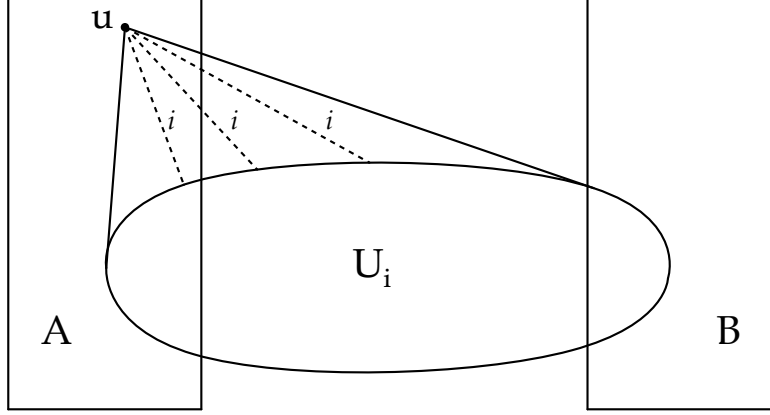


FIGURE 2. Case 1

If  $m \geq 3$ , we may assume that the result holds for  $m - 2$ . We then define  $\Delta' : ((U_i \cap A) \sqcup (U_j \cap B))^{(2)} \rightarrow \mathbb{N} \cup \{\mathbf{0}\}$  by

$$\Delta'(e) = \begin{cases} \Delta(e) & \text{if } \Delta(e) \notin \{i, j\} \\ \mathbf{0} & \text{if } \Delta(e) \in \{i, j\}. \end{cases}$$

Then, by the induction hypothesis, there is a subset  $X'$  of  $(U_i \cap A) \sqcup (U_j \cap B)$  such that  $X' \cap (U_i \cap A) \neq \emptyset$ ,  $X' \cap (U_j \cap B) \neq \emptyset$  and  $\gamma_{\Delta'}(X') = m - 2$ . Since  $X' \subset (U_i \sqcup U_j)$ , all the edges between  $u$  and  $X'$  are coloured either  $i$  or  $j$  and as  $X' \cap U_i \neq \emptyset$  and  $X' \cap U_j \neq \emptyset$ , edges of both colours are present. Since both colours  $i$  and  $j$  are not counted by  $\gamma_{\Delta'}$ , we have  $\gamma_{\Delta}(X' \cup \{u\}) = m$ . Clearly,  $X \cap A \neq \emptyset$  and  $X \cap B \neq \emptyset$  and therefore,  $X = X' \cup \{u\}$  is the required subset.

Now suppose that  $m = 2$ . Since  $\gamma(w) < \infty$  for all  $w \in \mathbb{N}$ , we can find an infinite matching  $M = \{a_1b_1, a_2b_2, \dots\}$  between  $U_i \cap A$  and  $U_j \cap B$  such that each edge  $a_kb_k$  has a distinct colour. Further, we may suppose that no edge  $a_kb_l$  is coloured  $\mathbf{0}$ , for otherwise  $X = \{u, a_k, b_l\}$  is immediately the required subset. Since  $\gamma(\{a_1, a_2, \dots\}) < \infty$ , by Ramsey's theorem, there exists a 1-coloured subset  $\{a'_1, a'_2, \dots\}$  of  $\{a_1, a_2, \dots\}$ . Let  $a'_k$  be matched to the vertex  $b'_k$  in  $M$  and let  $c_k$  denote the colour of the edge  $a'_kb'_k$ .

If the colour of  $\{a'_1, a'_2, \dots\}$  is  $\mathbf{0}$ , then since  $\gamma(a'_1) < \infty$ , there exist  $s, t \in \mathbb{N}$  such that  $a'_1b'_s$  and  $a'_1b'_t$  have the same colour, say  $d$ . By our choice of  $M$ ,  $c_s \neq c_t$ . Hence, at least one of  $c_s$  or  $c_t$ , say  $c_s$ , is not equal to  $d$ . Then, it is easy to check that  $X = \{a'_1, a'_s, b'_s\}$  is the required subset.

If the colour of  $\{a'_1, a'_2, \dots\}$  is  $d \neq \mathbf{0}$ , we may assume (by discarding the edge  $a'_1b'_1$  and relabeling the remaining vertices if necessary) that  $c_1$ , the colour of the edge  $a'_1b'_1$ , is not equal to  $d$ . Since  $\gamma(b'_1) < \infty$ , there exist  $s, t \in \mathbb{N}$  such that  $\Delta(a'_sb'_1) = \Delta(a'_tb'_1)$ . If  $\Delta(a'_sb'_1) = d$ , then



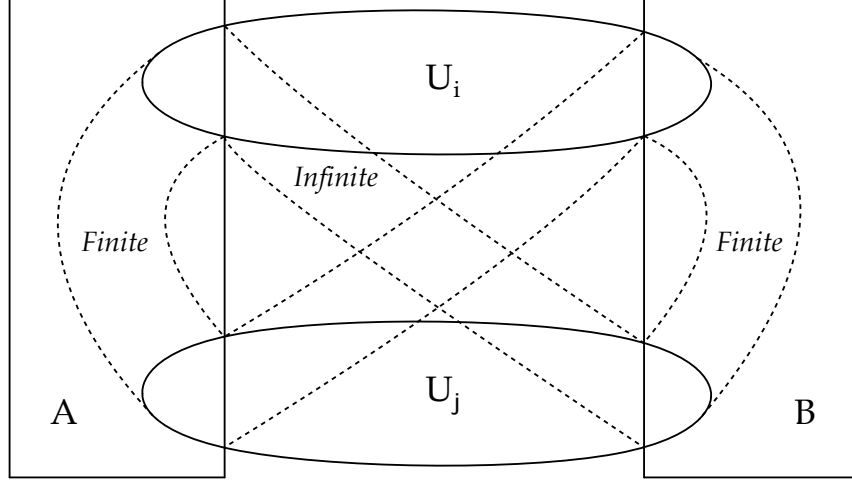


FIGURE 3. Case 2

we may take  $X = \{a'_1, a'_s, b'_1\}$ . On the other hand, if  $\Delta(a'_s b'_1) \neq d$ , then  $X = \{a'_s, a'_t, b'_1\}$  is the required subset.  $\square$

CASE 3:  $\gamma(U_0, U_i) = \infty$  for some  $i \neq 0$

We argue as we did in Case 2. We may assume that  $\gamma(U_0 \cap A, U_i \cap B) = \infty$ . Define  $\Delta' : ((U_0 \cap A) \sqcup (U_i \cap B))^{(2)} \rightarrow \mathbb{N} \cup \{0\}$  by

$$\Delta'(e) = \begin{cases} \Delta(e) & \text{if } \Delta(e) \neq i \\ 0 & \text{if } \Delta(e) = i. \end{cases}$$

By the induction hypothesis, there exists a subset  $X'$  of  $(U_0 \cap A) \sqcup (U_i \cap B)$  such that  $X' \cap (U_0 \cap A) \neq \emptyset$ ,  $X' \cap (U_i \cap B) \neq \emptyset$  and  $\gamma_{\Delta'}(X') = m - 1$ . As before, all the edges between  $u$  and  $X'$  are coloured either  $0$  or  $i$  and edges of both colours are present. Since colour  $i$  is not counted by  $\gamma_{\Delta'}$ , we have  $\gamma_{\Delta}(X' \cup \{u\}) = m$ . Hence,  $X = X' \cup \{u\}$  is the required subset.  $\square$

So, we may assume that  $\gamma(U_0) = \infty$ . Recall that we chose  $u \in A$  and  $v \in B$  such that  $\Delta(uv) = c \neq 0$ . Since  $\gamma(v) < \infty$ , we have a partition of  $U_0 = V_0 \sqcup V_1 \sqcup \dots \sqcup V_n$ , with  $n \leq \gamma(v)$ , based on the colour of the edge joining a given vertex of  $U_0$  to the vertex  $v$ . Applying the same argument as in Cases 1, 2 and 3 to the vertex  $v$ , we see that we are done unless  $\gamma(V_0) = \infty$ .

In this case, we consider the partition  $V_0 = (V_0 \cap A) \sqcup (V_0 \cap B)$ . We have  $\gamma(V_0 \cap A) < \infty$ ,  $\gamma(V_0 \cap B) < \infty$  and  $\gamma(V_0 \cap A, V_0 \cap B) = \infty$ . Define  $\Delta' : V_0^{(2)} \rightarrow \mathbb{N} \cup \{0\}$  by

$$\Delta'(e) = \begin{cases} \Delta(e) & \text{if } \Delta(e) \neq c \\ 0 & \text{if } \Delta(e) = c. \end{cases}$$

By the induction hypothesis, there is a subset  $X'$  of  $V_{\mathbf{0}}$  such that  $\gamma_{\Delta'}(X') = m - 1$ . Observe that  $uv$  has colour  $c$  and all the edges between  $\{u, v\}$  and  $X' \subset V_{\mathbf{0}} \subset U_{\mathbf{0}}$  have colour  $\mathbf{0}$ . Since the colour  $c$  is not counted by  $\gamma_{\Delta'}$ , we have  $\gamma_{\Delta}(X' \cup \{u, v\}) = m$ . Therefore,  $X = X' \cup \{u, v\}$  is the required subset since clearly,  $X \cap A \neq \emptyset$  and  $X \cap B \neq \emptyset$ . This completes the proof. ■

We are now in a position to deduce Theorem 6 from Lemma 7.

**3.2. Proof of Theorem 6.** We shall prove by induction on  $m$  that if  $\mathbb{N}$  has no rainbow coloured or star coloured infinite subset, then  $m \in \mathcal{G}_{\Delta}$ . The result is trivial for  $m = 1$ ; there exists an edge  $uv$  with a nonzero colour and thus  $\gamma(\{u, v\}) = 1$ . Now suppose that  $m \geq 2$ . We shall inductively find a subset  $X$  of  $\mathbb{N}$  with  $\gamma(X) = m$ .

If  $\gamma(v) = \infty$  for some vertex  $v \in \mathbb{N}$ , then we can find an infinite subset  $U = \{u_1, u_2, \dots\}$  of  $\mathbb{N}$  such that the edges  $vu_i$  and  $vu_j$  have distinct colours for all  $i \neq j$ . Applying Theorem 1 to the restriction of  $\Delta$  to  $U^{(2)}$ , we can find an infinite subset  $W = \{w_1, w_2, \dots\}$  of  $U$  such that  $W$  is either rainbow coloured, 1-coloured, left coloured, or right coloured.  $W$  cannot be rainbow coloured. If  $W$  is 1-coloured, it is clear that we can find an infinite subset  $W'$  of  $W$  (by removing at most one vertex from  $W$ ) such that  $W' \cup \{v\}$  is star coloured with centre  $v$ . If  $W$  is left or right coloured, then  $\gamma(\{w_1, \dots, w_{m+1}\}) = m$  unless all the edges incident with one of  $w_1, \dots, w_{m+1}$  are coloured  $\mathbf{0}$ , in which case  $\gamma(\{w_{m+2}, \dots, w_{2m+2}\}) = m$ .

So we may assume that  $\gamma(v) < \infty$  for all  $v \in \mathbb{N}$ . Pick an edge  $uv$  of nonzero colour, say  $c$ . We may suppose that the colours of the edges incident on  $u$  are  $\mathbf{0}, 1, \dots, \gamma(u)$ . We then partition

$$\mathbb{N} \setminus \{u\} = U_{\mathbf{0}} \sqcup U_1 \sqcup \dots \sqcup U_{\gamma(u)}$$

where  $U_i$  is the set of all vertices that are joined to  $u$  by an edge of colour  $i$ . Since  $\gamma(\mathbb{N}) = \infty$ , by the pigeonhole principle, we must either have  $\gamma(U_i) = \infty$  for some  $i$ , or  $\gamma(U_i, U_j) = \infty$  for some  $i \neq j$ . If  $\gamma(U_{\mathbf{0}}) < \infty$ , then we have the following two cases.

CASE 1:  $\gamma(U_i) < \infty$  for all  $i \neq \mathbf{0}$

Since  $\gamma(\mathbb{N}) = \infty$ , it must be the case that  $\gamma(U_i, U_j) = \infty$  for some  $i \neq j$ . Applying Lemma 7 to  $\Delta : (U_i \sqcup U_j)^{(2)} \rightarrow \mathbb{N} \cup \{\mathbf{0}\}$ , we find a subset  $X$  of  $U_i \sqcup U_j$  such that  $\gamma(X) = m$ . □

CASE 2:  $\gamma(U_i) = \infty$  for some  $i \neq \mathbf{0}$

Define a  $\mathbf{0}$ -infinite colouring  $\Delta' : U_i^{(2)} \rightarrow \mathbb{N} \cup \{\mathbf{0}\}$  by

$$\Delta'(e) = \begin{cases} \Delta(e) & \text{if } \Delta(e) \neq i \\ \mathbf{0} & \text{if } \Delta(e) = i. \end{cases}$$

The fact that  $\gamma_{\Delta}(w) < \infty$  for all  $w \in \mathbb{N}$  implies that  $\gamma_{\Delta'}(w) < \infty$  for all  $w \in U_i$ . So  $U_i$  has no rainbow or star coloured infinite subset with respect to  $\Delta'$ . By the induction hypothesis, there is a subset  $X'$  of  $U_i$  such that  $\gamma_{\Delta'}(X') = m - 1$ . Observe that all the edges between  $u$  and

$X' \subset U_i$  have colour  $i$ . Since the colour  $i$  is not counted by  $\gamma_{\Delta'}$ , we have  $\gamma_{\Delta}(X' \cup \{u\}) = m$ . Therefore,  $X = X' \cup \{u\}$  is the required subset.  $\square$

So, we may suppose that  $\gamma(U_0) = \infty$ . Recall that the edge  $uv$  has colour  $c \neq \mathbf{0}$ . Since  $\gamma(v) < \infty$ , we have a partition of  $U_0 = V_0 \sqcup V_1 \sqcup \cdots \sqcup V_n$ , with  $n \leq \gamma(v)$ , based on the colour of the edge joining a given vertex of  $U_0$  to the vertex  $v$ . Applying the same argument as in Cases 1 and 2 to the vertex  $v$ , we see that we are done unless  $\gamma(V_0) = \infty$ . In this case, we define  $\Delta' : V_0^{(2)} \rightarrow \mathbb{N} \cup \{\mathbf{0}\}$  by

$$\Delta'(e) = \begin{cases} \Delta(e) & \text{if } \Delta(e) \neq c \\ \mathbf{0} & \text{if } \Delta(e) = c. \end{cases}$$

The fact that  $\gamma_{\Delta}(w) < \infty$  for all  $w \in \mathbb{N}$  implies that  $\gamma_{\Delta'}(w) < \infty$  for all  $w \in V_0$ . So  $V_0$  has no rainbow or star coloured infinite subset with respect to  $\Delta'$ . By the induction hypothesis, there is a subset  $X'$  of  $V_0$  such that  $\gamma_{\Delta'}(X') = m - 1$ . Observe that  $uv$  has colour  $c$  and all the edges between  $\{u, v\}$  and  $X' \subset V_0 \subset U_0$  have colour  $\mathbf{0}$ . Since the colour  $c$  is not counted by  $\gamma_{\Delta'}$ , we have  $\gamma_{\Delta}(X' \cup \{u, v\}) = m$ . Therefore,  $X = X' \cup \{u, v\}$  is the required subset. This completes the proof.  $\blacksquare$

#### 4. EXTENSIONS AND APPLICATIONS

In this section, we shall first describe a finitary analogue of Theorem 2. We then use this to prove Theorems 4 and 5. Throughout this section, by a *countable set*, we will mean a set that is either finite or countably infinite.

**4.1. Finite Colourings.** We can prove a version of Theorem 2 for colourings (of finite or infinite complete graphs) that use only finitely many colours.

**Theorem 2'.** Let  $V$  be a countable set. Then, for all  $n \in \mathbb{N}$ , there exists a natural number  $k_1 = k_1(n)$  such that for every colouring  $\Delta : V^{(2)} \rightarrow \mathbb{N}$  which uses at least  $k_1$  distinct colours, either

- there is an  $m$ -coloured complete subgraph for every  $m \in [n]$ ; or
- there exists a rainbow coloured complete subgraph on  $n$  vertices; or
- there exists a star coloured complete subgraph on  $n$  vertices.

This result can be proved by arguments similar to those used to prove Theorem 2. There are two essential differences. First, as opposed to Theorem 1, we use an extension of the theorem, proved by Erdős and Rado, to colourings of *finite* complete graphs with an arbitrary set of colours. Second, in the place of Lemma 7, we use the following finitary analogue which is proved in the same way as the lemma.

**Lemma 7'.** Let  $V$  be a countable set. Then, for all  $m, d, D \in \mathbb{N}$ , there exists a natural number  $k_2 = k_2(m, d, D)$  with the following property:

For every colouring  $\Delta : V^{(2)} \rightarrow \mathbb{N} \cup \{\mathbf{0}\}$  such that

- $\gamma(v) < d$  for all  $v \in V$ , and
- there is a partition of  $V = A \sqcup B$  such that  $\gamma(A) < D$ ,  $\gamma(B) < D$  and  $\gamma(A, B) \geq k_2$ ,

there exists a subset  $X$  of  $V$  such that  $X \cap A \neq \emptyset$ ,  $X \cap B \neq \emptyset$  and  $\gamma(X) = m$ .

**4.2. Proof of Theorem 4.** Theorem 4 may be deduced from Theorem 2'. Recall that Theorem 4 says that the propositions  $F(\mathbb{N}, k, \binom{n}{2})$  and  $F(\mathbb{N}, k, \binom{n}{2} + 1)$  both hold provided that the number of colours  $k$  is sufficiently large. In fact, a little more is true; we do not need the vertex set to be  $\mathbb{N}$ . The following easy corollary of Theorem 2' shows that  $F(V, k, \binom{n}{2})$  holds for any  $V$  when  $k$  is sufficiently large.

**Corollary 8.** *Let  $V$  be a countable set. Then, for all  $n \in \mathbb{N}$ , there exists a natural number  $C_1 = C_1(n)$  such that  $F(V, k, \binom{n}{2})$  holds for all  $k \geq C_1$ .*

*Proof.* Take  $C_1(n) = k_1(\binom{n}{2})$ , where  $k_1$  is the constant guaranteed by Theorem 2'. ■

We can also show that  $F(V, k, \binom{n}{2} + 1)$  when  $k$  is sufficiently large, but in this case, we also need the set of vertices  $V$  to be large.

**Theorem 9.** *Let  $V$  be a countable set. Then, for all  $n \in \mathbb{N}$ , there exists a natural number  $C_2 = C_2(n)$ , and for all  $k \geq C_2$  there exists a natural number  $D_{k,n}$  such that  $F(V, k, \binom{n}{2} + 1)$  holds for each  $k \geq C_2$ , provided  $|V| \geq D_{k,n}$ .*

*Proof.* For  $n = 2$ , Stacey and Weidl noted that the result is true with  $C_2(2) = 2$  and  $D_{k,2} = R(k+1; k)$ , the Ramsey number for finding a monochromatic  $K_{k+1}$  when using  $k$  colours.

For  $n \geq 3$ , let  $s = n^4$ . We claim that  $C_2(n) = k_1(s)$  will do, where  $k_1$  is the constant guaranteed by Theorem 2'. For  $k \geq C_2(n)$ , we take  $D_{k,n} = k^s + s + 1$ . Now, suppose that  $\Delta : V^{(2)} \rightarrow [k]$  is a colouring that uses exactly  $k$  colours and that  $|V| \geq D_{k,n}$ . Then, by our choice of  $C_2(n)$ , either

- there is an  $l$ -coloured complete subgraph for every  $l \in [s]$ ; or
- there exists a rainbow coloured complete subgraph on  $s$  vertices; or
- there exists a star coloured complete subgraph on  $s$  vertices.

Note that a star coloured complete subgraph on  $s$  vertices contains an  $l$ -coloured complete subgraph for  $2 < l \leq s$ . Since  $2 < \binom{n}{2} + 1 \leq s$ , we are done unless there exists a rainbow coloured complete subgraph on  $s$  vertices. Hence, suppose that the complete subgraph on the vertex set  $S = \{u_1, u_2, \dots, u_s\}$  is rainbow coloured. For each  $x \in V \setminus S$ , there are  $k^s$  possible

values for the  $s$ -tuple  $(\Delta(xu_1), \Delta(xu_2), \dots, \Delta(xu_s))$ . Since,  $|V \setminus S| \geq D_{k,n} - s > k^s$ , we can find vertices  $x, y \in V \setminus S$  such that

$$(\Delta(xu_1), \Delta(xu_2), \dots, \Delta(xu_s)) = (\Delta(yu_1), \Delta(yu_2), \dots, \Delta(yu_s)).$$

We claim that there is a subset  $T \subset S$  of size  $t = n^2$  such that for all  $u \in T$ ,  $\Delta(xu) \notin \Delta(T^{(2)})$ . Assume for the sake of contradiction that for every subset  $T \subset S$  of size  $t$ , there exists at least one vertex  $u \in T$  such that  $\Delta(xu) \in \Delta(T^{(2)})$ . Consider the set

$$A = \{(u, T) : u \in T \subset S, |T| = t, \Delta(xu) \in \Delta(T^{(2)})\}.$$

By our assumption, for each  $T \subset S$  of size  $t$ , there is at least one  $u \in T$  such that  $(u, T) \in A$ , and so  $|A| \geq \binom{s}{t}$ . Since  $S$  is rainbow coloured, for each  $u \in S$ , there is at most one edge  $ab$  in  $S^{(2)}$  of colour  $\Delta(xu)$ . If  $(u, T)$  is in  $A$ , then we must have  $a, b \in T$ . So for each  $u \in S$ , there are at most  $\binom{s-2}{t-2}$  sets  $T$  such that  $(u, T) \in A$ . Thus,  $|A| \leq s \binom{s-2}{t-2}$ . Combining these two inequalities for  $|A|$ , we get

$$\binom{s}{t} \leq |A| \leq s \binom{s-2}{t-2}.$$

This means that  $t(t-1) \leq s-1$ , contradicting the fact that  $s = t^2$ .

Hence, there is indeed a subset  $T$  of  $S$  of size  $t = n^2$  such that for all  $u \in T$ , we have  $\Delta(xu) \notin \Delta(T^{(2)})$ . Consider  $\mathcal{C} = \{\Delta(xu) : u \in T\}$ . If  $|\mathcal{C}| < n$ , then as  $|T| = n^2$ , there are vertices  $v_1, v_2, \dots, v_n$  in  $T$  such that

$$\Delta(xv_1) = \Delta(xv_2) = \dots = \Delta(xv_n).$$

Since this colour  $\Delta(xv_1)$  is not an element of  $\Delta(T^{(2)})$ , we conclude that  $\{x, v_1, v_2, \dots, v_n\}$  is  $(\binom{n}{2} + 1)$ -coloured.

So we may assume that  $|\mathcal{C}| \geq n$ . Then there is a subset  $U \subset T$  of size  $n$  such that the colours  $\Delta(xu)$  are distinct for all  $u \in U$ . Since  $U \subset T$ , the colour  $\Delta(xu)$  is not an element of  $\Delta(U^{(2)})$  for each  $u \in U$ . We hence conclude that  $U \cup \{x\}$  is rainbow coloured.

Recall that there is a vertex  $y \neq x$  in  $V \setminus S$  such that  $\Delta(xu) = \Delta(yu)$  for all  $u \in S$ . Since at most one edge  $e$  in  $(U \cup \{x\})^{(2)}$  is coloured with the same colour as the edge  $xy$ , by removing the endpoint of  $e$  which lies in  $U$  if necessary, we can find a subset  $U'$  of  $U$  of size  $n-1$  such that  $\Delta(xy)$  is not an element of  $\Delta((U' \cup \{x\})^{(2)})$ . Then  $U' \cup \{x, y\}$  is  $(\binom{n}{2} + 1)$ -coloured since  $U' \cup \{x\}$  and  $U' \cup \{y\}$  are rainbow coloured sets of size  $n$  using the same set of colours. ■

It is easy to see that, taken together, Corollary 8 and Theorem 9 imply Theorem 4.

**4.3. Proof of Theorem 5.** Recall that Theorem 5 says that one can find an  $m$ -coloured complete bipartite subgraph when the complete bipartite graph between two copies of  $\mathbb{N}$  is coloured with sufficiently many colours. It turns out that the result remains true even if we

do not have infinitely many vertices; it is sufficient to have a large number of colours. The following corollary of Lemma 7' is slightly stronger than Theorem 5.

**Corollary 10.** *Let  $U$  and  $V$  be countable sets. Then, for all  $m \in \mathbb{N}$ , there exists a natural number  $C' = C'(m)$  such that if  $\Delta : U \times V \rightarrow \mathbb{N}$  is a colouring of the complete bipartite graph between  $U$  and  $V$  that uses at least  $C'$  distinct colours, then there exist  $X \subset U$  and  $Y \subset V$  such that the complete bipartite subgraph between  $X$  and  $Y$  is  $m$ -coloured.*

*Proof.* It suffices to take  $C'(m) = k_2(m, m, 1)$ , where  $k_2$  is the constant guaranteed by Lemma 7'.

To see this, define a colouring  $\Delta' : (U \cup V)^{(2)} \rightarrow \mathbb{N} \cup \{\mathbf{0}\}$  by

$$\Delta'(e) = \begin{cases} \mathbf{0} & \text{if } e \in U^{(2)} \text{ or } e \in V^{(2)} \\ \Delta(e) & \text{otherwise.} \end{cases}$$

If a vertex  $u \in U$  is incident with edges of at least  $m$  distinct colours, say  $uv_1, uv_2, \dots, uv_m$ , then we are done since we may take  $X = \{u\}$  and  $Y = \{v_1, v_2, \dots, v_m\}$ . Similarly, we are done if there is a vertex  $v \in V$  incident with edges of at least  $m$  distinct colours.

So, we may assume that  $\gamma_{\Delta'}(w) < m$  for all  $w \in U \cup V$ . But then, by our choice of  $C'$ , there exists a subset  $Z$  of  $U \cup V$  such that  $\gamma(Z) = m$ . In this case,  $X = Z \cap U$  and  $Y = Z \cap V$  are the required subsets. ■

## 5. CONCLUDING REMARKS

We conclude our paper by mentioning two questions that might merit further study. First, the problem of completely determining when the proposition  $F(V, k, m)$  holds is still open; while we have taken a few steps towards this in this paper, the full question is still far from being resolved. Second, it would be reasonable to ask the questions considered here for hypergraphs. However, even in the case of  $\mathbb{N}^{(3)}$ , it is not immediately clear to us what the canonical structures analogous to the rainbow coloured and star coloured complete graphs should be.

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